

On isomorphisms of a Brauer character ring onto another

著者	Yamauchi Kenichi
journal or publication title	Tsukuba journal of mathematics
volume	20
number	1
page range	207-212
year	1996-06
URL	http://hdl.handle.net/2241/6949

ON ISOMORPHISMS OF A BRAUER CHARACTER RING ONTO ANOTHER

Dedicated to Professor Hiroyuki Tachikawa

By

Kenichi YAMAUCHI

1. Introduction

Throughout this paper G , Z and Q denote a finite group, the ring of rational integers and the rational field respectively. Moreover we write \bar{Z} to denote the ring of all algebraic integers in the complex numbers and \bar{Q} to denote the algebraic closure of Q in the field of complex numbers. For a finite set S , we denote by $|S|$ the number of elements in S .

Let $\text{Irr}(G) = \{\chi_1, \dots, \chi_h\}$ be the complete set of absolutely irreducible complex characters of G . Then we can view χ_1, \dots, χ_h as functions from G into the complex numbers. We write $\bar{Z}R(G)$ to denote the \bar{Z} -algebra spanned by χ_1, \dots, χ_h . For two finite groups G and H , let λ be a \bar{Z} -algebra isomorphism of $\bar{Z}R(G)$ onto $\bar{Z}R(H)$. Then we can write

$$\lambda(\chi_i) = \sum_{j=1}^h a_{ij} \chi'_j, \quad (i = 1, \dots, h)$$

where $a_{ij} \in \bar{Z}$ and $\text{Irr}(H) = \{\chi'_1, \dots, \chi'_h\}$. In this case we write A to denote the $h \times h$ matrix with (i, j) -entry equal to a_{ij} and say that A is afforded by λ with respect to $\text{Irr}(G)$ and $\text{Irr}(H)$.

As is well known, concerning the isomorphism λ , we have the following two results, which seem to be most important. (For example see Theorem 1.3 (ii) and Lemma 3.1 in [5])

(i) $|c_G(c_i)| = |c_H(c'_i)|$, $(i = 1, \dots, h)$ where $\{c_1, \dots, c_h\}$ and $\{c'_1, \dots, c'_h\}$ are complete sets of representatives of the conjugate classes in G and H respectively and $c_i \xrightarrow{\lambda} c'_i$, $(i = 1, \dots, h)$. (Concerning a symbol " $c_i \xrightarrow{\lambda} c'_i$ ", see the definition in [5] and also the definition in section 2 in this paper)

(ii) A is unitary where A is the matrix afforded by λ with respect to $\text{Irr}(G)$ and $\text{Irr}(H)$.

In this paper our main objective is to give a necessary and sufficient condition

under which the above statements (i) and (ii) hold, concerning an isomorphism λ of a Brauer character ring onto another, and to state a generalization of theorems of Saksonov and Weidman about character tables of finite groups. (See Theorem 2, Corollary 2.1 in [3] and Theorem 3 in [4])

From now on, when we consider homomorphisms from an algebra to another, unless otherwise specified, we shall only deal with algebra homomorphisms.

2. Preliminaries

We fix a rational prime number p and use the following notation with respect to a finite group G .

G_o : the set of all p -regular elements of G

$Cl(G_o) = \{\mathfrak{C}_1 = \{1\}, \dots, \mathfrak{C}_r\}$: the complete set of p -regular conjugate classes in G

$\{c_1 = 1, \dots, c_r\}$: a complete set of representatives of $\mathfrak{C}_1, \dots, \mathfrak{C}_r$ respectively

$IBr(G) = \{\varphi_1 = 1, \dots, \varphi_r\}$: the complete set of irreducible Brauer characters of G , which can be viewed as functions from G_o into the complex numbers

For any subring R of the field of complex numbers such that $1 \in R$, we write $RBR(G)$ to denote the ring of linear combinations of $\varphi_1, \dots, \varphi_r$ over R . That is, $RBR(G)$ is the R -algebra spanned by $\varphi_1, \dots, \varphi_r$. In particular we use the notation $BR(G)$ instead of $ZBR(G)$ and say that $BR(G)$ is the Brauer character ring of G . Moreover we add the following notation.

$G(\overline{Q}/Q)$: the Galois group of \overline{Q} over Q

If $A = (a_{ij})$ is a matrix over \overline{Q} , then for $\sigma \in G(\overline{Q}/Q)$ we write A^σ to denote the matrix (a_{ij}^σ) . We use the common notation X^* for the conjugate transpose of a matrix X .

Now we define characteristic class functions on G_o .

DEFINITION 2.1. We define class functions f_i on G_o ($i = 1, \dots, r$) as follows

$$f_i(c_i) = 1, \quad f_i(c_j) = 0 \quad (i \neq j).$$

In this case we say that these class functions are the characteristic class functions on G_o and that f_i corresponds to \mathfrak{C}_i or \mathfrak{C}_i corresponds to f_i ($i = 1, \dots, r$).

Now we prove an easy lemma concerning characteristic class functions on G_o .

LEMMA 2.2. Let $\{f_1, \dots, f_r\}$ be the complete set of characteristic class functions on G_o . Then we have

$$f_i \in \overline{Q}BR(G), \quad (i = 1, \dots, r).$$

PROOF. Let \hat{f}_i be a characteristic class function of G such that $\hat{f}_i|_{G_o} = f_i$ where $\hat{f}_i|_{G_o}$ indicates the restriction of f_i to G_o . Then each \hat{f}_i is written as a \overline{Q} -linear combination of χ_1, \dots, χ_h . That is,

$$(2.1) \quad \hat{f}_i = \sum_{j=1}^h (|\mathfrak{G}_i|/|G|) \overline{\chi_j(c_i)} \chi_j, \quad (i = 1, \dots, r)$$

For each absolutely irreducible complex character χ_i of G , $\chi_i|_{G_o}$ is written as a \mathbb{Z} -linear combination of $\varphi_1, \dots, \varphi_r$. That is,

$$(2.2) \quad \chi_i|_{G_o} = \sum_{j=1}^r d_{ij} \varphi_j, \quad (i = 1, \dots, h)$$

where (d_{ij}) is the decomposition matrix of G .

By virtue of the formulas (2.1) and (2.2), we can conclude that $f_i \in \overline{Q}BR(G)$, $(i = 1, \dots, r)$ as required. Q.E.D.

We are given two finite groups G and H . For G and H we assume that there exists an isomorphism λ of $\overline{Z}BR(G)$ onto $\overline{Z}BR(H)$. Then it follows that the rank of $BR(G)$ = the rank of $BR(H)$ and $|Cl(G_o)| = |Cl(H_o)|$. We also can extend λ to an isomorphism $\hat{\lambda}$ of $\overline{Q}BR(G)$ onto $\overline{Q}BR(H)$ by linearity. By Lemma 2.2 we have $f_i \in \overline{Q}BR(G)$. Here we use the following additional notation.

$$Cl(H_o) = \{\mathfrak{G}'_1 = \{1\}, \dots, \mathfrak{G}'_r\}$$

$\{c'_1 = 1', \dots, c'_r\}$: a complete set of representatives of $\mathfrak{G}'_1, \dots, \mathfrak{G}'_r$ respectively

$\{f'_1, \dots, f'_r\}$: the complete set of characteristic class functions on H_o where f'_i corresponds to \mathfrak{G}'_i , $(i = 1, \dots, r)$.

$$IBr(H) = \{\varphi'_1 = 1, \dots, \varphi'_r\}.$$

We now show a lemma which is actually the key step in the proof of Lemma 2.4.

LEMMA 2.3. In the above situation, $\hat{\lambda}(f_i)$ is a characteristic class function on H_o , $(i = 1, \dots, r)$.

PROOF. Since $\overline{Q}BR(G)f_i = \overline{Q}f_i \cong \overline{Q}$, $\overline{Q}BR(G)f_i$ is a minimal ideal of $\overline{Q}BR(G)$ and so f_i is a (central) primitive idempotent, $(i = 1, \dots, r)$. Since $\hat{\lambda}(f_i) \in \overline{Q}BR(H)$, we can write

$$(2.3) \quad \hat{\lambda}(f_i) = \sum_{j=1}^r a_j f'_j, \quad a_j \in \overline{Q}$$

Since $f_i^2 = f_i$ and $f_i f'_j = 0$ ($i \neq j$), by the formula (2.3) we have

$$\hat{\lambda}(f_i) = \sum_{j=1}^r a_j^2 f'_j.$$

Thus $a_j^2 = a_j$, $(j = 1, \dots, r)$. Hence $a_j = 0$ or $a_j = 1$, $(j = 1, \dots, r)$. It follows that $\hat{\lambda}(f_i) = f'_j$ for some $j \in \{1, \dots, r\}$, because f_i is a primitive idempotent, hence the result. Q.E.D.

Now we define a bijection from $Cl(G_o)$ to $Cl(H_o)$ through the isomorphism λ as follows. For a p -regular conjugate class \mathfrak{C}_i of G , \mathfrak{C}_i corresponds to a characteristic class function f_i on G_o . Since by Lemma 2.3 $\hat{\lambda}(f_i)$ is also a characteristic class function $f'_{i''}$ on H_o , $\hat{\lambda}(f_i) = f'_{i''}$ corresponds to a p -regular conjugate class $\mathfrak{C}'_{i''}$ of H . Here we assign $\mathfrak{C}'_{i''}$ to \mathfrak{C}_i ($i = 1, \dots, r$). Thus we get a one-to-one correspondence between $Cl(G_o)$ and $Cl(H_o)$:

$$c_i \in \mathfrak{C}_i \rightarrow f_i \rightarrow \hat{\lambda}(f_i) = f'_{i''} \rightarrow \mathfrak{C}'_{i''} \ni c'_{i''}$$

where $i \rightarrow i''$ ($i = 1, \dots, r$) is a permutation. In this case we write $\mathfrak{C}_i \xrightarrow{\lambda} \mathfrak{C}'_{i''}$ or $c_i \xrightarrow{\lambda} c'_{i''}$ ($i = 1, \dots, r$).

Keeping the above notation, we give the following lemma concerning the Brauer character table of G . This lemma plays a fundamental role in the proof of Theorem 3.1. The proof is the same as that of Theorem 2.2 in [5] and so we omit its proof.

LEMMA 2.4. $(\varphi_i(c_j)) = (\lambda(\varphi_i)(c'_{j''}))$ ($r \times r$ matrices) where $c_j \xrightarrow{\lambda} c'_{j''}$, $(j = 1, \dots, r)$.

3. Main theorems

Let G and H be two finite groups with Cartan matrices C and C' respectively. Let λ be an isomorphism of $\overline{ZBR}(G)$ onto $\overline{ZBR}(H)$ and $A = (a_{ij})$ be the matrix afforded by λ with respect to $IBr(G) = \{\varphi_1, \dots, \varphi_r\}$ and $IBr(H) = \{\varphi'_1, \dots, \varphi'_r\}$. We set $Cl(G_o) = \{\mathfrak{C}_1, \dots, \mathfrak{C}_r\}$ and $Cl(H_o) = \{\mathfrak{C}'_1, \dots, \mathfrak{C}'_r\}$ and assume that $c_i \in \mathfrak{C}_i$, $c'_i \in \mathfrak{C}'_i$ and $c_i \xrightarrow{\lambda} c'_{i''}$ where $i \rightarrow i''$ ($i = 1, \dots, r$) is a permutation. We write \mathbf{m} to denote the vector with i -th entry equal to $|C_G(c_i)|$ and \mathbf{m}' to denote the vector with i -th entry equal to $|C_H(c'_{i''})|$, ($i = 1, \dots, r$). Then we have the following two theorems.

THEOREM 3.1. *With the above notation, $\mathbf{m} = \mathbf{m}'$ iff $A^*CA = C'$. This necessarily happens if $CA = AC'$, in which case A is clearly unitary.*

PROOF. To prove this theorem, we introduce some simplifying notation: Write P to denote the $r \times r$ matrix with (i, j) -entry equal to $\varphi_i(c_j)$ and similarly write P' for the matrix with (i, j) -entry equal to $\varphi'_i(c'_{j''})$.

Since $\lambda(\varphi_i) = \sum_{k=1}^r a_{ik} \varphi'_k$ where $A = (a_{ij})$, by Lemma 2.4 we have

$$\varphi_i(c_j) = \lambda(\varphi_i)(c'_{j'}) = \sum_{k=1}^r a_{ik} \varphi'_k(c'_{j'}).$$

This implies that $P = AP'$. Also, if B is the diagonal matrix with (i, i) -entry equal to $|C_G(c_i)|$, it follows that $P^*CP = B$ by Theorem 60.5 in [2]. Similarly $(P')^*C'P' = B'$, where B' is the diagonal matrix with (i, i) -entry equal to $|C_H(c'_{i'})|$. Here we note that $B = B'$ iff $m = m'$. Since $P^* = (P')^*A^*$, we have the two equations

$$(P')^*A^*CAP' = B \quad \text{and} \quad (P')^*C'P' = B'.$$

It is now obvious that $B = B'$ iff $A^*CA = C'$.

Now suppose $CA = AC'$. Then we show that A is unitary. If we write $J = A^*A$, then we have $(P')^*JC'P' = B$. Thus $(B')^{-1}B = (P')^{-1}(C')^{-1}JC'P'$. This is a diagonal matrix with rational entries and this shows that J has rational eigenvalues. But J has algebraic integer entries, and so must have integer eigenvalues. Thus $(B')^{-1}B$ is a diagonal matrix with positive integer diagonal entries. Also, A is invertible over $\bar{\mathbb{Z}}$ and thus A^* is too. It follows that $\det(J) = \det((B')^{-1}B) = 1$ and so $(B')^{-1}B$ is the identity matrix I . It follows that $J = A^*A = I$ and so A is unitary, as required. Q.E.D.

THEOREM 3.2. *If $CA = AC'$, then we have*

- (i) $\lambda(\varphi_i) = \varepsilon_i \varphi'_{i'}$ where the ε_i are roots of 1 and $i \rightarrow i'$ ($i = 1, \dots, r$) is a permutation.
- (ii) *The Brauer character tables of G and H are the same.*

PROOF. (i) Now we pay attention to the fact that if $\alpha \in \bar{\mathbb{Z}}$ and $|\alpha^\sigma| \leq 1$ (an absolute value) for all $\sigma \in G(\bar{\mathbb{Q}}/Q)$, then $\alpha = 0$ or α is a root of 1.

If we use the same notation as in the proof of Theorem 3.1, then we have $A = P(P')^{-1}$ and so A has entries that lie in a field with an abelian Galois group. Thus $(A^*)^\sigma = (A^\sigma)^*$ for all $\sigma \in G(\bar{\mathbb{Q}}/Q)$. Since A is unitary by Theorem 3.1, A^σ is automatically unitary for all $\sigma \in G(\bar{\mathbb{Q}}/Q)$. Hence we have the equation with respect to the i -th row of A^σ .

$$\sum_{j=1}^r a_{ij}^\sigma \overline{a_{ij}^\sigma} = \sum_{j=1}^r |a_{ij}^\sigma|^2 = 1, \quad (i = 1, \dots, r)$$

Hence we have $|a_{ij}^\sigma| \leq 1$ for all $\sigma \in G(\bar{\mathbb{Q}}/Q)$. This implies that $a_{ij} = 0$ or a_{ij} is a root of 1 because of the above attention. Thus it follows that for each $i \in \{1, \dots, r\}$, there exists $i' \in \{1, \dots, r\}$ such that $a_{ii'}$ is a root of 1 and $a_{ij} = 0$ ($j \neq i'$). Hence $\lambda(\varphi_i) = \varepsilon_i \varphi'_{i'}$ where $\varepsilon_i = a_{ii'}$ is a root of 1 and $i \rightarrow i'$ ($i = 1, \dots, r$) is a permutation.

- (ii) We state a one-to-one correspondence μ between $IBr(G)$ and $IBr(H)$

through the isomorphism λ as follows. By (i) of this theorem, we have $\lambda(\varphi_i) = \varepsilon_i \varphi'_i$ ($i = 1, \dots, r$) where the ε_i are roots of 1. Here we assign φ'_i to $\varphi_i : \mu(\varphi_i) = \varphi'_i$ ($i = 1, \dots, r$). Then μ can be extended to an isomorphism of $BR(G)$ onto $BR(H)$ by linearity. (See the proof of Lemma 3.2 in [5]) By Lemma 2.4 we have $(\varphi_i(c_j)) = (\varphi'_i(c'_{j''}))$ ($r \times r$ matrices) where $c_j \xrightarrow{\mu} c'_{j''}$ ($j = 1, \dots, r$). That is, G and H have the same Brauer character table. Thus the result follows. Q.E.D.

REMARK. If the condition $m = m'$ in Theorem 3.1 holds, then we can easily prove $|G| = |H|$. But we can give examples such that for two finite groups G, H with $|G| \neq |H|$, a matrix A is unitary where A is afforded by an isomorphism of $BR(G)$ onto $BR(H)$. Actually, such an example is given by taking G and H to be any two p -groups of different orders. Another example can be found in [1]. ($p = 2, G =$ the symmetric group S_4 on 4 symbols and $H =$ the dihedral group D_6 of order 12. See the examples of section 91 in [1])

Acknowledgement

The author would like to thank the referee for his valuable comments, especially about the greatly simplified proof of Theorem 3.1 of the paper.

References

- [1] Curtis, C. W. and Reiner, I., Representation theory of finite groups and associative algebras, Interscience, New York, 1962.
- [2] Dornhoff, L., Group Representation Theory, (two volumes), Dekker, New York, 1972.
- [3] Saksonov, A. I., The integral ring of characters of a finite group, (Russian), Vesci Akad. Navuk BSSR Ser. Fiz. -Mat. Navuk, 1966, No. 3, 69–76.
- [4] Weidman, D. R., The character ring of a finite group, Illinois, J. Math. **9** (1965), 462–467.
- [5] Yamauchi, K., On automorphisms of a character ring, preprint.

Department of Mathematics
Faculty of Education
Chiba University
Chiba-city 263
Japan